

SUPERSINGULAR K3 SURFACES ARE UNIRATIONAL

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ABSTRACT. We show that supersingular K3 surfaces are related by purely inseparable isogenies. As an application, we deduce that they are unirational, which confirms conjectures of Artin, Rudakov, Shafarevich, and Shioda. To complete the picture, we prove Shioda–Inose type “sandwich” theorems for K3 surfaces of Picard rank ≥ 19 in positive characteristic.

1. INTRODUCTION

The Picard rank ρ of a complex K3 surface satisfies $\rho \leq 20$. In [SI77], [I78], Shioda and Inose classified complex K3 surfaces with Picard rank 20, so-called *singular K3 surfaces*. They showed that such a surface rationally dominates and is rationally dominated by a Kummer surface, that is, it forms a “Kummer sandwich”. Moreover, they showed that singular K3 surfaces can be defined over number fields, and thus, form a countable set and have no moduli. Later, Shioda [Sh06] gave explicit constructions, and Ma [Ma13] gave a purely Hodge theoretic description. Morrison [Mo84] generalized the Shioda–Inose theorem to complex K3 surfaces with large Picard rank. Closely related to these results is Shafarevich’s conjecture according to which every Hodge-isogeny between the transcendental lattices of two complex K3 surfaces is induced by a rational map or a rational correspondence – we refer to Section 2.2 for details.

The first result of this article is an extension of the Shioda–Inose theorem to positive characteristic:

Theorem. *Let X be a K3 surface in odd characteristic with Picard rank 19 or 20. Then, there exists an ordinary Abelian surface A and dominant, rational maps*

$$\mathrm{Km}(A) \dashrightarrow X \dashrightarrow \mathrm{Km}(A),$$

both of which are generically finite of degree 2.

Our proof uses canonical Serre–Tate lifts and the Shioda–Inose theorem over the complex numbers. We refer to Theorem 2.6 for precise statements, fields of definition, as well as lifting results. For example, we show that a surface with Picard rank 20 can be defined over a finite field, and so, these surfaces form a countable set and have no moduli, also in positive characteristic.

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Artin [Ar74a] noted that there do not exist K3 surfaces with Picard rank 21 in any characteristic. On the other hand, Tate [Ta65] and Shioda [Sh77b] gave examples of K3 surfaces with Picard rank 22 in positive characteristic, so-called *Shioda-supersingular K3 surfaces*. Artin [Ar74a] showed that Shioda-supersingular K3 surfaces are *Artin-supersingular*, that is, their formal Brauer groups are of infinite height. It follows from recent progress in the Tate-conjecture for K3 surfaces due to Charles [Ch12], Madapusi Pera [MP13], and Maulik [Ma12] that a K3 surface in odd characteristic is Artin-supersingular if and only if it is Shioda-supersingular.

Artin [Ar74a] also showed that supersingular K3 surfaces form 9-dimensional families, which is in contrast to the above mentioned rigidity of singular K3 surfaces. Moreover, Shioda [Sh77b] showed that Tate's and his examples are unirational, another property of K3 surfaces that can happen in positive characteristic only. Since unirational K3 surfaces are supersingular as shown by Shioda [Sh74], this led several people to conjecture the converse:

Conjecture (Artin, Rudakov, Shafarevich, Shioda). A K3 surface is supersingular if and only if it is unirational.

This conjecture was established by Shioda [Sh77b] for supersingular Kummer surfaces in odd characteristic, by Rudakov and Shafarevich [RS78] in characteristic 2 and for K3 surfaces with Artin invariant $\sigma_0 \leq 6$ in characteristic 3, as well as for K3 surfaces with Artin invariant $\sigma_0 \leq 3$ in characteristic 5 by Pho and Shimada [PS06]. In particular, there do exist supersingular K3 surfaces that are unirational in every positive characteristic.

The key result of this article is a structure theorem for supersingular K3 surfaces, which was posed as an open question by Rudakov and Shafarevich in [RS78], and which is similar to the Shioda–Inose theorem for singular K3 surfaces.

Theorem. *Let X and X' be supersingular K3 surfaces with Artin invariants σ_0 and σ'_0 , respectively, in characteristic $p \geq 5$.*

- (1) *There exist dominant and rational maps*

$$X \dashrightarrow X' \dashrightarrow X,$$

which are purely inseparable and generically finite of degree $p^{2\sigma_0+2\sigma'_0-4}$.

- (2) *Let E be a supersingular elliptic curve. Then, there exist dominant and rational maps*

$$\mathrm{Km}(E \times E) \dashrightarrow X \dashrightarrow \mathrm{Km}(E \times E),$$

which are purely inseparable and generically finite of degree $p^{2\sigma_0-2}$.

In [SI77], [I78], Shioda and Inose introduced a notion of *isogeny* for singular K3 surfaces over the complex numbers, which was extended to other types of complex K3 surfaces by Morrison [Mo84] and Nikulin [Ni91]. We refer to Section 2.2 for an extension of this notion to positive characteristic, and in this terminology, our structure theorem says that all supersingular K3 surfaces are *purely inseparably isogenous*.

Our theorem also fits into Shafarevich’s conjecture already mentioned above: supersingular K3 surfaces are precisely those K3 surfaces without transcendental cycles in their second ℓ -adic cohomology. Thus, their “transcendental lattices” should be thought of as being zero, thus mutually isogenous, and by our theorem, they are all related by rational maps. We refer to Section 2.2 for details.

Our theorem also explains why supersingular K3 surfaces form 9-dimensional families, whereas singular K3 surface have no moduli: in both cases, these surfaces are isogenous to Kummer surfaces. For singular K3 surfaces, the isogeny is separable, and these rational maps do not deform. For supersingular K3 surfaces, the isogeny is purely inseparable, and these rational maps come in families. We refer to Remark 5.2 for details.

The main idea to proving this theorem is that a Jacobian elliptic fibration on a supersingular K3 surface with Artin invariant σ_0 gives rise to a one-dimensional deformation, such that all fibers in this family are elliptic supersingular K3 surfaces that are torsors under this Jacobian fibration. Moreover, the generic fiber of this family has Artin invariant $\sigma_0 + 1$ and is related to the special fiber by a rational and purely inseparable map of degree p^2 . Using Ogus’ moduli spaces [Og83], we then conclude that every K3 surface of Artin invariant $\sigma_0 + 1$ is purely inseparably isogenous to one of Artin invariant σ_0 . Thus, by induction on the Artin invariant, we obtain our theorem. We refer to Theorem 4.3 for a geometric picture and to Section 3 for details.

As already mentioned, supersingular Kummer surfaces in odd characteristic are unirational. Combined with our structure theorem for supersingular K3 surfaces, this establishes the Artin–Rudakov–Shafarevich–Shioda conjecture:

Theorem. *Supersingular K3 surfaces in characteristic $p \geq 5$ are unirational.*

Combined with results of Artin, Shioda, and the Tate-conjecture for K3 surfaces in odd characteristic, we obtain the following equivalence.

Theorem. *For a K3 surface X in characteristic $p \geq 5$, the following conditions are equivalent:*

- (1) *X is unirational.*
- (2) *The Picard rank of X is 22.*
- (3) *The formal Brauer group of X is of infinite height.*
- (4) *For all i , the F -crystal $H_{\text{cris}}^i(X/W)$ is of slope $i/2$.*

We refer to Section 3.4, Section 4.1, and Section 5.3 for partial results if $p \leq 3$. For example, the previous theorem also holds for $p = 3$ once the Rudakov–Shafarevich theorem [RS82] on potential good reduction of supersingular K3 surfaces is established in characteristic 3.

This article is organized as follows:

In Section 2, after reviewing formal Brauer groups, several notions of supersingularity, and introducing purely inseparable isogenies, we classify K3 surfaces with Picard ranks 19 and 20 in odd characteristic, which generalizes the classical Shioda–Inose theorem.

In Section 3, we show how a supersingular K3 surface with Artin invariant σ_0 together with a Jacobian elliptic fibration gives rise to a one-dimensional family of supersingular K3 surfaces that are torsors under this Jacobian fibration and whose generic fiber has Artin invariant $\sigma_0 + 1$. Moreover, we show how these torsors are related to the trivial torsor by purely inseparable rational maps.

In Section 4, we interpret these one-dimensional families in terms of Ogus' moduli spaces of supersingular K3 crystals. As an interesting byproduct, we find that these moduli spaces are related to each other by (iterated) \mathbb{P}^1 -bundles, together with a moduli interpretation of this structure. In particular, this gives a new description of these moduli spaces.

In Section 5, we use the results of Section 3 to show that supersingular K3 surfaces are related by purely inseparable isogenies. As an immediate corollary, we deduce their unirationality.

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2. NON-SUPERSINGULAR K3 SURFACES WITH LARGE PICARD NUMBER

In this section, we first review the formal Brauer group, and discuss several notions of supersingularity for K3 surfaces. Then, we classify non-supersingular K3 surfaces with large Picard rank in positive characteristic, which gives a structure result similar to the Shioda–Inose theorem over the complex numbers.

2.1. Formal Brauer groups, supersingularity, and Picard ranks. Let X be a K3 surface over a field k of positive characteristic. By results of Artin and Mazur [AM77], the functor on local Artinian k -algebras defined by

$$S \mapsto \ker \left(H_{\text{ét}}^2(X \times_k S, \mathbb{G}_m) \rightarrow H_{\text{ét}}^2(X, \mathbb{G}_m) \right)$$

is pro-representable by a one-dimensional formal group law $\widehat{\text{Br}}(X)$, the so-called *formal Brauer group*. Over algebraically closed fields, such group laws are determined by their *height*, and Artin [Ar74a, Theorem (0.1)] showed that the height h of the formal Brauer group of a K3 surface satisfies $1 \leq h \leq 10$ or $h = \infty$.

Definition 2.1. Let X be a K3 surface over a field of positive characteristic and let h be the height of its formal Brauer group. Then, X is called *ordinary* if $h = 1$, and X is called *Artin-supersingular* if $h = \infty$.

The general picture is as follows: a smooth and projective variety X over a perfect field of positive characteristic is called *ordinary*, if the Hodge- and Newton-polygons on all its crystalline cohomology groups coincide. It is called *supersingular* if the Newton-polygons on all its crystalline cohomology groups are straight lines, that is, if the F -crystal $H_{\text{cris}}^i(X/W)$ is of slope $i/2$ for all i . Now, if X is a K3 surface, this general definition translates into a condition on $H_{\text{cris}}^2(X/W)$ only. More precisely, being ordinary translates into having slopes $(0, 1, 2)$, and

being supersingular into being of slope 1. By [II79, Section II.7.2], the Frobenius-slopes on $H_{\text{cris}}^2(X/W)$ in terms of the height h of the formal Brauer group are $(1 - \frac{1}{h}, 1, 1 + \frac{1}{h})$. Thus, for K3 surfaces, Definition 2.1 coincides with the general definition.

For surfaces, Shioda [Sh74] introduced another notion of supersingularity. To explain it, we note that the first Chern class map $c_1 : \text{NS}(X) \rightarrow H^1(\Omega_X^1)$ is injective over the complex numbers, which implies that the Picard rank ρ of a smooth projective variety is bounded above by $h^1(\Omega_X^1)$. For complex K3 surfaces, this gives the estimate $\rho \leq 20$. In positive characteristic, Igusa [Ig60] established the weaker inequality $\rho \leq b_2$, which, for K3 surfaces, gives the estimate $\rho \leq 22$. And indeed, Tate [Ta65] and Shioda [Sh77b] showed that there do exist K3 surfaces with Picard rank 22 in positive characteristic.

Definition 2.2. Let X be a K3 surface over an algebraically closed field. Then, X is called *singular* if $\rho = 20$, and it is called *Shioda-supersingular* if $\rho = 22$.

The relation between these two notions of supersingularity is as follows: In [Ar74a, Theorem (0.1)], Artin showed that a K3 surface whose formal Brauer group is of finite height h satisfies $\rho \leq b_2 - 2h$. This implies that Shioda-supersingular K3 surfaces are Artin-supersingular. In [Ar74a, Theorem (4.3)], Artin proved that Artin-supersingular K3 surfaces that are elliptic are Shioda-supersingular. In general, the equivalence of Artin- and Shioda-supersingularity follows from the Tate-conjecture for K3 surfaces. Since this has been recently established in odd characteristic by Charles [Ch12], Madapusi Pera [MP13], and Maulik [Ma12], let us summarize these results as follows.

Theorem 2.3 (Artin, Charles, Madapusi Pera, Maulik, et al.). *For a K3 surface X in odd characteristic, the following are equivalent:*

- (1) X is Shioda-supersingular, that is, $\rho = 22$.
- (2) X is Artin-supersingular, that is, $h(\widehat{\text{Br}}(X)) = \infty$.
- (3) For all i , the F -crystal $H_{\text{cris}}^i(X/W)$ is of slope $i/2$. □

By [Ar74a, Section 4], the discriminant of the Néron–Severi lattice of a Shioda-supersingular K3 surface is equal to $-p^{2\sigma_0}$ for some integer $1 \leq \sigma_0 \leq 10$.

Definition 2.4. The integer σ_0 is called the *Artin-invariant* of X .

The Artin invariant σ_0 gives rise to a stratification of the moduli space of Shioda-supersingular K3 surfaces [Ar74a, Section 7], and it determines the Néron–Severi lattice of a Shioda-supersingular K3 surface up to isometry [RS78, Section 1]. We refer the interested reader to the overview articles by Shioda [Sh79] and Rudakov–Shafarevich [RS81] for basic properties of Shioda-supersingular K3 surfaces, details and further references.

2.2. Isogenies between K3 surfaces. For Abelian varieties, the notion of isogeny is classical. For K3 surfaces, there are several and conflicting extensions of this notion, and we refer to [Mo87, Section 1] for an overview. Following Inose [I78], we use the most naive one, which is sufficient for the purposes of this article.

Definition 2.5. Let X and Y be varieties of the same dimension over a perfect field of positive characteristic p . An *isogeny of degree n* from X to Y is a dominant, rational, and generically finite map $X \dashrightarrow Y$ of degree n . A *purely inseparable isogeny of height h* is an isogeny that is purely inseparable of degree p^h .

For Abelian varieties A, B and an isogeny $A \rightarrow B$, there exists an integer n such that multiplication by $n : A \rightarrow A$ factors through this isogeny. Such a factorization gives rise to an isogeny $B \rightarrow A$, and in particular, being isogenous is an equivalence relation. Over the complex numbers, K3 surfaces with Picard rank 20 are related to Kummer surfaces by isogenies, and the existence of an isogeny in the other direction is a true, but non-trivial fact, see [SI77] and [I78].

Coming back to Definition 2.5, if $X \dashrightarrow Y$ is a purely inseparable isogeny of height h , the h -fold Frobenius $F^h : X \rightarrow X$ factors through this isogeny, inducing an isogeny $Y \dashrightarrow X$, which is purely inseparable of height $(d-1)h$, where d is the dimension of X and Y . In particular, being purely inseparable isogenous is an equivalence relation.

Since it motivates some of our results later on and sheds another light on them, let us shortly discuss a conjecture of Shafarevich concerning complex K3 surfaces: let X and Y be complex K3 surfaces with transcendental lattices $T(X)$ and $T(Y)$. If $\rho(X) = \rho(Y) = 20$, then $T(X)$ and $T(Y)$ are of rank 2, and the Shioda–Inose theorem [SI77] implies that every isogeny $T(X) \rightarrow T(Y)$ preserving Hodge structures induces and is induced by an isogeny between the corresponding surfaces. These results were generalized by Morrison [Mo84] and Nikulin [Ni87], [Ni91] to K3 surfaces with higher rank transcendental lattices, and Shafarevich conjectured that Hodge isogenies between transcendental lattices of complex K3 surfaces are always induced by isogenies, or, rational correspondences. Here, the right definition of isogeny for K3 surfaces is one difficulty, and we refer to [Mo87, Section 1] for discussion and the relation of Shafarevich’s conjecture to the Hodge conjecture. Let us also note that results of Chen [Ch10] imply that Shafarevich’s conjecture cannot be true if one only allows isogenies in the sense of our naive Definition 2.5.

In positive characteristic, a K3 surface X is Shioda-supersingular if and only if every class in $H_{\text{ét}}^2(X, \mathbb{Q}_\ell)$ is algebraic if and only if the cokernel of $c_1 : \text{NS}(X) \rightarrow H_{\text{cris}}^2(X/W)$ is a W -module that is torsion. (In fact, the length of $\text{coker}(c_1)$ is the Artin invariant.) Therefore, the “transcendental lattices” of Shioda-supersingular K3 surfaces should be thought of as being zero, in which case, they would all be isogenous for trivial reasons. Now, if one boldly believes in a characteristic- p version of Shafarevich’s conjecture (even whose precise formulation is unclear to the author at the moment), one might expect that all Shioda-supersingular K3 surfaces are related by isogenies. This was posed as Question 8 by Rudakov and Shafarevich at the end of [RS78], and we shall prove it in Theorem 5.1 below.

2.3. The Shioda–Inose theorem in odd characteristic. In this subsection, we classify non-supersingular K3 surfaces with Picard rank $\rho \geq 19$ in odd characteristic, which is an analog of the Shioda–Inose theorem [SI77], [I78] over the complex numbers. The idea in positive characteristic is to show first that such surfaces are

ordinary, which implies that they possess canonical lifts to the Witt ring, namely, Serre–Tate lifts. Then, we use the Shioda–Inose theorem in characteristic zero to deduce a similar structure result in positive characteristic.

Theorem 2.6. *Let X be a K3 surface with Picard rank $19 \leq \rho \leq 21$ over an algebraically closed field k of characteristic $p \geq 3$. Then,*

- (1) *X is an ordinary K3 surface, and*
- (2) *X lifts together with its Picard group projectively to $\mathrm{Spec} W(k)$.*

Moreover,

- (3) *If $\rho = 19$, then there exists an ordinary Abelian surface A over k , and isogenies of degree 2*

$$\mathrm{Km}(A) \dashrightarrow X \dashrightarrow \mathrm{Km}(A).$$

Moreover, neither X nor A can be defined over a finite field.

- (4) *If $\rho = 20$, then there exist two ordinary and isogenous elliptic curves E and E' over k , and isogenies of degree 2*

$$\mathrm{Km}(E \times E') \dashrightarrow X \dashrightarrow \mathrm{Km}(E \times E').$$

Moreover, X can be defined over a finite field. The lift of $(X, \mathrm{Pic}(X))$ is unique and coincides with the canonical Serre–Tate lift of X .

- (5) *The case $\rho = 21$ does not exist.*

Remark 2.7. Non-existence of K3 surfaces with Picard rank 21 was already observed by Artin [Ar74a, p. 544].

PROOF. First, let us show claims (1) and (5): let h be the height of the formal Brauer group. Since $\rho \geq 5$, X is elliptic, and since $\rho < 22$, it follows from [Ar74a, Theorem 1.7] that $h < \infty$. In particular, the formula $\rho \leq b_2 - 2h \leq 20$ from [Ar74a, Theorem 0.1] implies that $\rho = 21$ is impossible. The same formula shows that if $19 \leq \rho \leq 20$, then we must have $h = 1$, that is, X is ordinary.

Next, we show claim (2): since $h < \infty$, there exists a lift $\mathcal{X} \rightarrow \mathrm{Spf} W(k)$ of the pair $(X, \mathrm{Pic}(X))$, see the discussion in [LM11, Section 4] and in particular, [LM11, Corollary 4.2]. By loc. cit. this lift is unique if $\rho = 20$ and since the canonical Serre–Tate lift of X also has the property that $\mathrm{Pic}(X)$ lifts, the two lifts coincide. In any case, since there is an ample invertible sheaf among the lifted ones, \mathcal{X} is algebraizable by Grothendieck’s existence theorem [Il05, Theorem 8.4.10].

Now, we show claim (4): the idea is to start with a lift $X \rightarrow \mathcal{X}$ of $(X, \mathrm{Pic}(X))$ to characteristic zero. Then, we apply the classical Shioda–Inose theorem to the geometric generic fiber $\mathcal{X}_{\overline{K}}$, and show that there exists a model \mathcal{Y} of $\mathcal{X}_{\overline{K}}$ with good reduction, whose reduction also satisfies the conclusion of claim (4). By the Matsusaka–Mumford theorem, this reduction is isomorphic to X , thereby establishing claim (4). Thus, let us assume $\rho = 20$, and let \mathcal{X} be a lift of $(X, \mathrm{Pic}(X))$ as asserted by claim (2). We denote by K the field of fractions of $W(k)$. By construction, the geometric generic fiber $\mathcal{X}_{\overline{K}}$ of $\mathcal{X} \rightarrow \mathrm{Spec} W(k)$ is an algebraic K3 surface with $\rho = 20$. By the classical Shioda–Inose theorem from [SI77] and [I78] (but see also [Ma13, Theorem 2.5]), there exist isogenous elliptic curves \tilde{E} and \tilde{E}'

with complex multiplication over \overline{K} , and a symplectic involution ι on the Kummer surface $\mathrm{Km}(\tilde{E} \times \tilde{E}')$, such that $\mathcal{X}_{\overline{K}}$ is the desingularization of the quotient $\mathrm{Km}(\tilde{E} \times \tilde{E}')/\langle \iota \rangle$. Since elliptic curves with complex multiplication have potential good reduction, after possibly passing to a finite extension $R \supseteq W(k)$, there exists a model of $\mathrm{Km}(\tilde{E} \times \tilde{E}')$ over R with good reduction, which is actually itself a Kummer surface $\mathrm{Km}(\mathcal{E} \times \mathcal{E}')$ (here, we also use that $p \neq 2$, so that the quotient by the sign involution can be formed over R without trouble). After possibly enlarging R again, the involution ι is defined on the generic fiber $\mathrm{Km}(\mathcal{E} \times \mathcal{E}')_K$. Now, ι extends to an involution on $\mathrm{Km}(\mathcal{E} \times \mathcal{E}')$, see, for example the proof of [LM11, Theorem 2.1]. Since the involution acts trivially on the global 2-form of the generic fiber, it will also act trivially on the global 2-form of the special fiber. In particular, ι extends to a symplectic involution on $\mathrm{Km}(\mathcal{E} \times \mathcal{E}') \rightarrow \mathrm{Spec} R$. On the geometric generic fiber $\mathrm{Km}(\mathcal{E} \times \mathcal{E}')_{\overline{K}}$, the symplectic involution ι has precisely 8 fixed points by [Ni80] or [Mo84, Lemma 5.2]. The same is true for the induced involution on the special fiber $\mathrm{Km}(\mathcal{E} \times \mathcal{E}')_k$ by [DK09, Theorem 3.3] (here, we use again $p \neq 2$). Thus, after possibly enlarging R again, we may form the quotient $\mathrm{Km}(\mathcal{E} \times \mathcal{E}')/\langle \iota \rangle$ and resolve the resulting 8 families of A_1 -singularities to get a smooth family $\mathcal{Y} \rightarrow \mathrm{Spec} R$. After possibly enlarging R again, the generic fibers \mathcal{Y}_K and \mathcal{X}_K become isomorphic. Since both models, \mathcal{X} and \mathcal{Y} , have good reduction, and their special fibers are not ruled, their special fibers are isomorphic by the Matsusaka–Mumford theorem [MM64, Theorem 2]. This shows the existence of rational dominant map $\mathrm{Km}(E \times E') \dashrightarrow X$, which is generically finite of degree 2. Here, E and E' are the reductions of \mathcal{E} and \mathcal{E}' , respectively. The existence of a rational dominant map $X \dashrightarrow \mathrm{Km}(E \times E')$, generically finite of degree 2, follows from the corresponding characteristic zero statement as before and we leave it to the reader. Since $h = 1$, Frobenius acts bijectively on $H^2(X, \mathcal{O}_X)$, from which we conclude that it also acts bijectively on $H^2(\mathrm{Km}(E \times E'), \mathcal{O}_{\mathrm{Km}(E \times E')})$, as well as on $H^2(E \times E', \mathcal{O}_{E \times E'})$. In particular, $E \times E'$ is an ordinary Abelian surface, that is, E and E' are ordinary. (Alternatively, one can also argue via their formal Brauer groups as in [DK09, Lemma 4.4].) Finally, since \tilde{E} and \tilde{E}' are elliptic curves with complex multiplication, they can be defined over $\overline{\mathbb{Q}}$. In particular, \mathcal{E} , \mathcal{E}' , $\mathrm{Km}(\mathcal{E} \times \mathcal{E}')$ and ι can be defined over $W(\overline{\mathbb{F}}_p)$, and we thus obtain a model of X over $\overline{\mathbb{F}}_p$.

Finally, we sketch a proof of (3): we assume $\rho = 19$, and as before, we lift $(X, \mathrm{Pic}(X))$ to some $\mathcal{X} \rightarrow \mathrm{Spec} W(k)$. By the above mentioned classification results, there exists an Abelian variety \tilde{A} over \overline{K} and an involution ι on $\mathrm{Km}(\tilde{A})$ such that $\mathrm{Km}(\tilde{A})/\iota$ is isomorphic to the geometric generic fiber $\mathcal{X}_{\overline{K}}$. Since \mathcal{X} has good reduction at p , the Galois-action of the absolute Galois group $G_K := \mathrm{Gal}(\overline{K}/K)$ on $H_{\mathrm{ét}}^2(X, \mathbb{Q}_{\ell})$, $\ell \neq p$, is unramified. From the explicit description, it follows that also the G_K -actions on $H_{\mathrm{ét}}^2(\mathrm{Km}(\tilde{A}), \mathbb{Q}_{\ell})$ and $H_{\mathrm{ét}}^2(\tilde{A}, \mathbb{Q}_{\ell})$ are unramified. Thus, by the Néron–Ogg–Shafarevich criterion, \tilde{A} has good reduction, that is, there exists a smooth model $\mathcal{A} \rightarrow \mathrm{Spec} W(k)$, whose special fiber A is an Abelian surface. As in the case of $\rho = 20$, we find rational dominant maps $\mathrm{Km}(A) \dashrightarrow X$ and

$X \dashrightarrow \mathrm{Km}(A)$, both of which are generically finite of degree 2. Inspecting the Frobenius-actions on H^2 (or the formal Brauer groups) as above, we conclude that A is an ordinary Abelian surface. Finally, since X is elliptic, the Tate conjecture holds for X , and thus, if X were definable over $\overline{\mathbb{F}}_p$, its geometric Picard rank would have to be even [Ar74a, p. 544]. This implies that X , as well as A and $\mathrm{Km}(A)$, cannot be defined over $\overline{\mathbb{F}}_p$. \square

Remark 2.8. We would like to point out the following analogy between zero and positive characteristic for K3 surfaces with Picard rank 20, that is, singular K3 surfaces: over the complex numbers, such surfaces can be defined over $\overline{\mathbb{Q}}$, and thus, form a countable set and have no moduli. In characteristic $p \geq 3$, such surfaces can be defined over $\overline{\mathbb{F}}_p$, and again, form a countable set and have no moduli.

3. CONTINUOUS FAMILIES OF TORSORS

In this section, we show that a supersingular K3 surface X with Artin invariant σ_0 that is equipped with a Jacobian elliptic fibration admits a one-dimensional deformation, whose generic fiber has Artin invariant $\sigma_0 + 1$, and whose generic fiber is related to the special fiber X by a purely inseparable isogeny (in the sense of Definition 2.5).

In order to avoid confusion later on, let us fix the following terminology.

Definition 3.1. A fibration from a surface onto a curve is called *elliptic* if its generic fiber is smooth of genus one. In case this fibration admits a section, we will call it a *Jacobian elliptic fibration*, and a choice of section, referred to as the *zero section*, is part of the data of a Jacobian elliptic fibration.

3.1. Families of torsors arising from formal Brauer groups. In this subsection, we will closely follow the setup of the articles [AS73] and [Ar74a] by Artin and Swinnerton-Dyer. Let $f : X \rightarrow \mathbb{P}^1$ be a Jacobian elliptic fibration, where X is a K3 surface over an algebraically closed field k . Associated to this fibration we have its Weierstraß model $f' : X' \rightarrow \mathbb{P}^1$, which is obtained by contracting those (-2) -curves in the fibers of f that do not meet the zero section. If f has reducible fibers, then X' has rational double point singularities. We denote by $A \subseteq X'$ the smooth locus of X' . As explained in [AS73, Section 1], A is a group scheme over \mathbb{P}^1 . In terms of Néron models, the smooth locus of X over \mathbb{P}^1 is the Néron model of its generic fiber, and then, A is the identity component.

Next, let $S \rightarrow \mathrm{Spec} k$ be a (possibly formal) scheme over k together with a section $0 : \mathrm{Spec} k \rightarrow \mathrm{Spec} S$. We want to classify families of torsors under A , parametrized by S , such that the special fiber is the trivial A -torsor, that is, we

consider Cartesian diagrams

$$(1) \quad \begin{array}{ccc} A & \longrightarrow & \overline{A} \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 \times_k S \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & S \end{array}$$

In order to understand such “moving torsors”, we compute $H_{\text{ét}}^2(X', \mathbb{G}_m)$ using the Grothendieck–Leray spectral sequence

$$E_2^{p,q} := H_{\text{ét}}^p(\mathbb{P}^1, R^q f'_* \mathbb{G}_m) \implies H_{\text{ét}}^{p+q}(X', \mathbb{G}_m).$$

As explained in [Ar74a, Section 2], the formal structure of $H_{\text{ét}}^2(X', \mathbb{G}_m)$ is that of $H_{\text{ét}}^1(\mathbb{P}^1, \text{Pic}_{X'/\mathbb{P}^1})$. Using the zero section, we identify $\text{Pic}_{X'/\mathbb{P}^1}^0$ with $A \rightarrow \mathbb{P}^1$, and see that there is a relation between families of A -torsors as in Diagram (1) and the formal Brauer group of X' . The next proposition makes this precise.

Proposition 3.2. *Let $X \rightarrow \mathbb{P}^1$ be a Jacobian elliptic fibration on a K3 surface over an algebraically closed field k of positive characteristic p , and let $A \rightarrow \mathbb{P}^1$ be the smooth locus of the Weierstraß fibration $X' \rightarrow \mathbb{P}^1$. Let $S := \text{Spf } R$, where (R, \mathfrak{m}_R) is a local, Noetherian, and complete k -algebra. Let $n \geq 1$ be an integer.*

- (1) *Formal families of A -torsors $\overline{A} \rightarrow \mathbb{P}^1 \times_k S$, together with a degree- n multisection D , and whose special fiber is the trivial A -torsor, are classified by the Abelian group*

$$\mathcal{K}_{n,A}(S) := \ker \left(H_{\text{ét}}^1(\mathbb{P}^1 \times_k S, {}_n A) \xrightarrow{\text{res}} H_{\text{ét}}^1(\mathbb{P}^1, {}_n A) \right),$$

where ${}_n A$ denotes the n -torsion subgroup scheme of A , and where res denotes the restriction map.

- (2) *If p does not divide n , then $\mathcal{K}_{n,A}(S) = 0$.*
(3) *If $p = n$, then $D \rightarrow \mathbb{P}^1 \times_k S$ is purely inseparable of degree p .*
(4) *There exists a natural and functorial isomorphism of Abelian groups*

$$\mathcal{K}_{n,A}(S) \cong \widehat{{}_n \text{Br}}(X)(S).$$

PROOF. We start with claim (1). To simplify notation, we shall use a subscript S to denote the trivial product family over S . If $Y = \mathbb{P}_k^1$ or $Y = \mathbb{P}_S^1$, then elements of $H_{\text{ét}}^1(Y, {}_n A)$ are in bijection to A -torsors over Y together with a degree- n multisection [AS73, Proposition (1.7)]. In particular, families of A -torsors over S , whose special fiber is the trivial A -torsor, correspond to elements of $H_{\text{ét}}^1(\mathbb{P}_S^1, {}_n A)$ that lie in the kernel of the restriction map to $H_{\text{ét}}^1(\mathbb{P}^1, {}_n A)$ and thus, are in bijection to elements of $\mathcal{K}_{n,A}(S)$.

Next, let us show claim (4). As explained in [AS73, Section 1], $H_{\text{ét}}^2(X'_S, \mu_n)$ can be computed via a Grothendieck–Leray spectral sequence, and the edge homomorphism

$$H_{\text{ét}}^2(X'_S, \mu_n) \longrightarrow H^0(\mathbb{P}_S^1, R^2 f'_* \mu_n) \cong \mathbb{Z}/n\mathbb{Z}$$

can be interpreted as the restriction of a cohomology class to one (any) geometric fiber. Also, still following [AS73, Section 1], and using the zero section, we obtain a splitting of the canonical homomorphism

$$H_{\text{fl}}^2(\mathbb{P}_S^1, \mu_n) \longrightarrow H_{\text{fl}}^2(X'_S, \mu_n).$$

By [AS73, Proposition (1.4)], the group $H_{\text{fl}}^1(\mathbb{P}_S^1, {}_nA)$ is canonically isomorphic to the subgroup of $H_{\text{fl}}^2(\mathbb{P}_S^1, \mu_n)$ of elements orthogonal to the fibers and the zero section of X'_S , respectively. Next, we consider the commutative diagram with exact rows, whose vertical maps are restriction maps

$$\begin{array}{ccccc} 0 & \rightarrow & H_{\text{fl}}^1(\mathbb{P}_S^1, {}_nA) & \rightarrow & H_{\text{fl}}^2(X'_S, \mu_n) \\ & & \downarrow & & \downarrow \text{res}(S) \\ 0 & \rightarrow & H_{\text{fl}}^1(\mathbb{P}^1, {}_nA) & \rightarrow & H_{\text{fl}}^2(X', \mu_n) \end{array}$$

We have an inclusion $\mathcal{K}_{n,A}(S) \subseteq \ker(\text{res}(S))$ by definition of $\mathcal{K}_{n,A}(S)$. To show equality, let $\xi \in \ker(\text{res}(S))$, that is, ξ restricts to zero in the special fiber. In particular, ξ is orthogonal to the zero section and the geometric fibers of X'_S , which shows that ξ lies in $\mathcal{K}_{n,A}(S)$. Thus, we have shown that there is an isomorphism

$$(2) \quad \mathcal{K}_{n,A}(S) \cong \ker \left(H_{\text{fl}}^2(X'_S, \mu_n) \xrightarrow{\text{res}} H_{\text{fl}}^2(X', \mu_n) \right).$$

Next, we recall from [AS73, Section 1] that there are isomorphisms $R^1 f'_* \mu_n \cong {}_nA$, and $R^2 f'_* \mu_n \cong \mathbb{Z}/n\mathbb{Z}$, as well as $R^1 f'_* \mathbb{G}_m \cong \text{Pic}_{X'/Y} \cong A \oplus \mathbb{Z}_Y$. Taking flat cohomology in the Kummer sequence $0 \rightarrow \mu_n \rightarrow \mathbb{G}_m \rightarrow \mathbb{G}_m \rightarrow 0$, we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc} \text{Pic}(X'_S)/n & \rightarrow & H_{\text{fl}}^2(X'_S, \mu_n) & \rightarrow & H_{\text{fl}}^2(X'_S, \mathbb{G}_m) & \rightarrow & H_{\text{fl}}^2(X'_S, \mathbb{G}_m) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \text{Pic}(X')/n & \rightarrow & H_{\text{fl}}^2(X', \mu_n) & \rightarrow & H_{\text{fl}}^2(X', \mathbb{G}_m) & \rightarrow & H_{\text{fl}}^2(X', \mathbb{G}_m) \end{array}$$

whose vertical arrows are restriction maps. By definition of the formal Picard group, we have $\widehat{\text{Pic}}(X')(S) = \ker(\text{Pic}(X'_S) \rightarrow \text{Pic}(X'))$, which is zero since $\text{Pic}_{X'/k}^0$ is trivial. Using the section $0 \rightarrow S$, we see that the restriction $\text{Pic}(X'_S) \rightarrow \text{Pic}(X')$ is surjective and thus, an isomorphism. In particular, $\text{Pic}(X'_S)/n \rightarrow \text{Pic}(X')/n$ is an isomorphism, from which we conclude that the commutative diagram above gives rise to an isomorphism

$$\ker \left(H_{\text{fl}}^2(X'_S, \mu_n) \xrightarrow{\text{res}} H_{\text{fl}}^2(X', \mu_n) \right) \cong {}_n\widehat{\text{Br}}(X')(S).$$

Combined with the isomorphism (2) and the isomorphism $\widehat{\text{Br}}(X') \cong \widehat{\text{Br}}(X)$ from [Ar74a, Proposition (2.1)], we get claim (4).

It remains to establish claims (2) and (3). Restricting a non-trivial family of A -torsors $\overline{A} \rightarrow \mathbb{P}_S^1$ as described in claim (1) to its degree- n multisection D , we obtain a family $D \rightarrow \mathbb{P}_S^1$ of torsors under a finite flat group scheme G_n of length n . More precisely, G_n is the subgroup scheme of ${}_nA$ that stabilizes D , and we remind the reader that ${}_nA$ is generically étale over \mathbb{P}^1 if and only if p does not divide n . By our setup, this family of G_n -torsors is trivial over $0 \in S$ and thus, if G_n were generically étale over \mathbb{P}^1 , the whole family would be trivial by Hensel's lemma (R is local and complete), thus trivializing the family $\overline{A} \rightarrow \mathbb{P}_S^1$ of A -torsors. This

contradiction shows that families of A -torsors as described in claim (1) do not exist if p and n are coprime, and we conclude $\mathcal{K}_{n,A}(S) = 0$ in this case. If $p = n$, this contradiction implies that $D \rightarrow \mathbb{P}_S^1$ must be purely inseparable. \square

3.2. Families of supersingular K3 surfaces. We recall from Section 2.1 that the formal Brauer group of a K3 surface is a one-dimensional formal group law, and that over algebraically closed fields, the only invariant of such a formal group law is its height. In view of Proposition 3.2.(4), we want to understand the n -torsion of the formal Brauer group, and start with the following observation, which holds for one-dimensional formal group laws in general.

Lemma 3.3. *Let G be a one-dimensional formal group law of height h over an algebraically closed field k of positive characteristic p . Let (R, \mathfrak{m}_R) be a local, Noetherian, and complete k -algebra. Let $n \geq 1$ be an integer.*

(1) *If $h = \infty$, then $G \cong \widehat{\mathbb{G}}_a$, and we have*

$${}_pG(R) \cong \mathfrak{m}_R, \quad \text{as well as} \quad {}_nG(R) = 0 \text{ if } p \text{ does not divide } n.$$

(2) *If $h < \infty$ and R is reduced, then ${}_nG(R) = 0$ for all n .*

PROOF. If p does not divide n , then multiplication by n is an isomorphism, and we obtain ${}_nG(R) = 0$.

If $h = \infty$, then $G \cong \widehat{\mathbb{G}}_a$ and thus, there is an isomorphism $\widehat{\mathbb{G}}_a(R) \cong \mathfrak{m}_R$ of Abelian groups. In particular, we have ${}_pG(R) \cong \mathfrak{m}_R$.

If $h < \infty$, then, by definition, the h -fold Frobenius factors through multiplication by p . If R is reduced, then Frobenius is injective, and thus, multiplication by p is injective in G . In particular, we find ${}_pG(R) = 0$. \square

This lemma shows how the formal Brauer group controls families of torsors as considered in Proposition 3.2.(1): first, the degree of the multisection cannot be prime to p (which we saw already in Proposition 3.2.(2)), and second, if we want to have a non-trivial family over a reduced base, then the K3 surface must be Artin-supersingular. This renders precise Artin's remark: "The unusual phenomenon of continuous families of homogeneous spaces occurs only for supersingular surfaces" [Ar74a, footnote (2) on p. 552].

The next proposition explores the geometry of such families of torsors, and is the key to Theorem 4.3 and Theorem 5.1 below.

Proposition 3.4. *Let $X \rightarrow \mathbb{P}^1$ be a Jacobian elliptic fibration on a supersingular K3 surface. Then, there exists a projective family of supersingular elliptic K3 surfaces with non-trivial moduli*

$$\overline{X} \rightarrow \mathbb{P}_S^1 \rightarrow S, \quad \text{where} \quad S := \operatorname{Spec} k[[t]],$$

whose central fiber over $0 \in S$ is $X \rightarrow \mathbb{P}^1$. Moreover,

(1) *The Artin invariant of the geometric generic fiber $\overline{X}_{\overline{K}}$ satisfies*

$$\sigma_0(\overline{X}_{\overline{K}}) = \sigma_0(X) + 1.$$

(2) *There exist purely inseparable isogenies of height 1 over $K := k((t))$*

$$\begin{array}{ccc} & Y & \\ \swarrow & & \searrow \\ X_K & & \overline{X}_K \end{array}$$

In particular, X_K and \overline{X}_K are related by a rational and purely inseparable correspondence.

(3) *There exist purely inseparable isogenies of height 2*

$$X_K \dashrightarrow \overline{X}_K \dashrightarrow X_K,$$

whose composition is twice the K -linear Frobenius morphism.

PROOF. As before, we denote by $X' \rightarrow \mathbb{P}^1$ the Weierstraß model of $X \rightarrow \mathbb{P}^1$, and by $A \subseteq X'$ its smooth locus. Since X is Artin-supersingular, we have that ${}_p\widehat{\mathrm{Br}}(X)(k[[t]])$ is non-zero by Lemma 3.3. By Proposition 3.2, there exists a non-trivial formal family of A -torsors

$$\overline{A} \rightarrow \mathbb{P}_S^1 \rightarrow S,$$

whose special fiber is the trivial A -torsor.

First, let us show that this formal family is algebraizable: we extend the given compactification $A \subseteq X'$ of the special fiber to some compactification $\overline{A} \rightarrow \overline{X}'$. Next, let $D \rightarrow \mathbb{P}_S^1$ be the degree- p multisection that comes with this family, let E be the class of a fiber, and set $\mathcal{L}_n := \mathcal{O}_{\overline{X}'}(D + nE)$. For $n \gg 0$, the restriction of \mathcal{L}_n to the special fiber X' is a Cartier divisor with positive self-intersection. Since $X' \rightarrow \mathbb{P}^1$ has irreducible fibers, every integral curve on X' is either a fiber, or D , or has positive intersection with a fiber and non-negative intersection with D . Thus, for $n \gg 0$, the restriction of \mathcal{L}_n to X' has positive self-intersection, and positive intersection with every integral curve on X' , and is thus ample by the Nakai–Moishezon criterion [Kl66, Chapter III.1]. Therefore, the formal family \overline{X}' is algebraizable by Grothendieck's existence theorem [Il05, Theorem 8.4.10].

Next, let us desingularize the singular compactification \overline{X}' : since this family of elliptic fibrations is a family of A -torsors, the singular fibers do not change their type [CD89, Theorem 5.3.1], and thus, $\overline{X}' \rightarrow \mathbb{P}_S^1$ is an equisingular family with rational double points. Blowing up the singularities in families, we can pass to a simultaneous resolution of singularities and obtain the desired smooth compactification

$$\overline{X} \rightarrow \overline{X}' \rightarrow \mathbb{P}_S^1 \rightarrow S.$$

Next, let us show claims (2) and (3). The family $\overline{X} \rightarrow \mathbb{P}_S^1$ comes with a degree- p multisection D . In particular, the base change $D \rightarrow \mathbb{P}_S^1$ trivializes this family of A -torsors, that is, we obtain a birational map

$$X_D \xrightarrow{\cong} \overline{X} \times_{\mathbb{P}_S^1} D.$$

(In fact, we have an isomorphism $A_D \cong \overline{A} \times_{\mathbb{P}_S^1} D$. However, it is unclear whether this isomorphism extends to the smooth compactification \overline{X} . Therefore, we only

claim the existence of rational map that is fiberwise a birational rational map.) By Proposition 3.2.(3), the morphism $D \rightarrow \mathbb{P}_S^1$ is purely inseparable of degree p . Thus, we obtain a diagram

$$(3) \quad X_S \longleftarrow X_D \xrightarrow{\cong} \overline{X} \times_{\mathbb{P}_S^1} D \longrightarrow \overline{X},$$

where the morphisms on the left and right are purely inseparable of degree p . This gives a description of \overline{X} in terms of purely inseparable correspondences, and we get claim (2). Moreover, the S -linear Frobenius morphism $X_S \rightarrow X_S$ factors through $X_D \rightarrow X_S$, and we obtain

$$X_S \dashrightarrow \overline{X} \times_{\mathbb{P}_S^1} D \longrightarrow \overline{X},$$

whose composition is purely inseparable of degree p^2 over S . Passing to generic fibers, we get claim (3).

Finally, in order to show claim (1) on Artin invariants, we use that specialization of invertible sheaves induces an injective homomorphism of Abelian groups

$$(4) \quad \mathrm{NS}(\overline{X}_{\overline{K}}) \longrightarrow \mathrm{NS}(X),$$

where $\overline{X}_{\overline{K}}$ denotes the geometric generic fiber. We want to show that the cokernel of this map is cyclic of order p , generated by the class of the zero section of the Jacobian fibration on X . This then implies that the Artin invariants of X and $\overline{X}_{\overline{K}}$ differ by one. From [AS73, (2.2)], we obtain a commutative diagram with exact rows, whose vertical arrows are restriction maps:

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathrm{Pic}(\mathbb{P}_{\overline{K}}^1) & \rightarrow & \mathrm{Pic}(\overline{X}'_{\overline{K}}) & \rightarrow & H^0(\mathbb{P}_{\overline{\eta}}^1, \mathrm{Pic}_{\overline{X}'_{\overline{K}}/\mathbb{P}_{\overline{K}}^1}) \rightarrow 0 \\ & & \uparrow & & \uparrow \rho_1 & & \uparrow \\ 0 & \rightarrow & \mathrm{Pic}(\mathbb{P}_S^1) & \rightarrow & \mathrm{Pic}(\overline{X}') & \rightarrow & H^0(\mathbb{P}_S^1, \mathrm{Pic}_{\overline{X}'/\mathbb{P}_S^1}) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \mathrm{Pic}(\mathbb{P}^1) & \rightarrow & \mathrm{Pic}(X') & \rightarrow & H^0(\mathbb{P}^1, \mathrm{Pic}_{X'/\mathbb{P}^1}) \rightarrow 0 \end{array}$$

Replacing S by a finite flat cover if necessary, we may assume that ρ_1 is an isomorphism, and then, all upward arrows will be isomorphisms. It follows from [AS73, Proposition (1.6)] that there is a commutative diagram of group algebraic spaces over $\mathbb{P}_{\overline{K}}^1$, \mathbb{P}_S^1 and \mathbb{P}_k^1 , respectively:

$$\begin{array}{ccccccc} 0 & \rightarrow & A_{\overline{K}} & \rightarrow & \mathrm{Pic}_{\overline{X}'_{\overline{K}}/\mathbb{P}_{\overline{K}}^1} & \rightarrow & \underline{\mathbb{Z}}_{\mathbb{P}_{\overline{K}}^1} \rightarrow 0 \\ & & \uparrow & & \uparrow \rho_2 & & \uparrow \\ 0 & \rightarrow & A_S & \rightarrow & \mathrm{Pic}_{\overline{X}'/\mathbb{P}_S^1} & \rightarrow & \underline{\mathbb{Z}}_{\mathbb{P}_S^1} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & A & \rightarrow & \mathrm{Pic}_{X'/\mathbb{P}^1} & \rightarrow & \underline{\mathbb{Z}}_{\mathbb{P}^1} \rightarrow 0 \end{array}$$

where the surjective morphisms in the rows are given by associating an invertible sheaf its degree on the fibers. Moreover, the zero section defines a splitting of the bottom row [AS73, (1.2)]. Since ρ_1 is an isomorphism, ρ_2 induces an isomorphism on global sections. Thus, taking global sections in the latter commutative diagram, we find that $\mathrm{Pic}(\overline{X}'_{\eta})$ is a subgroup of $\mathrm{Pic}(X')$, whose cokernel is a cyclic group. Next, $\mathrm{Pic}(X)$ is generated by the exceptional divisors of the contraction morphism

$\nu : X \rightarrow X'$ and $\nu^*\text{Pic}(X')$, and similarly for $\text{Pic}(\overline{X}_{\overline{K}})$. We have already seen above that X' and $\overline{X}'_{\overline{K}}$ have the same singularities, and deduce from this that also $\text{NS}(X)$ and $\text{NS}(\overline{X}_{\overline{K}})$ differ by a cyclic group. By [Ar74a, Corollary (1.3)], this cyclic group is of order p or trivial. However, it cannot be trivial for the following reason: let Z be the zero section of the Jacobian fibration $X \rightarrow \mathbb{P}^1$. We have $Z^2 = -\chi(\mathcal{O}_X) = -2$, and $h^0(\mathcal{O}_X(Z)) = 1$, $h^1(\mathcal{O}_X(Z)) = h^2(\mathcal{O}_X(Z)) = 0$ by Riemann–Roch and Serre duality. If $\mathcal{O}_X(Z)$ extended to $\overline{X}_{\overline{K}}$, then $Z^2 = -2$ and Riemann–Roch would imply that this extension is still effective. But this would imply that the zero section extends to the whole family of A -torsors, and would trivialize it, contradicting the fact that this family is non-trivial. Thus, $\mathcal{O}_X(Z)$ does not lie in the image of the specialization homomorphism (4) and its cokernel is therefore non-trivial of order p , generated by the class of Z . In particular, we obtain $\sigma_0(\overline{X}_{\overline{K}}) = \sigma_0(X) + 1$, which establishes claim (1). Since the Artin invariants of the geometric fibers differ, the family has non-constant moduli. \square

In characteristic $p \geq 5$, supersingular K3 surfaces do not degenerate, that is, have potential good reduction, by a theorem of Rudakov and Shafarevich [RS82], we conclude that the family over $k[[t]]$ described in the previous proposition arises from a *smooth* family of supersingular K3 surfaces over a smooth and *proper* curve.

Corollary 3.5. *If $p \geq 5$ and under the assumptions of Proposition 3.4, there exist a smooth and projective curve C over k , a closed point $0 \in C$, and a smooth and projective family of supersingular K3 surfaces*

$$\overline{Y} \rightarrow C$$

such that

- (1) $\overline{X} \rightarrow S$ is the fiber over the completed local ring $\widehat{\mathcal{O}}_{C,0}$. In particular, X is the fiber over 0.
- (2) If the fiber over a closed point of C has Artin invariant $\sigma_0(X) + 1$, then it is related to X by a purely inseparable isogeny of height 2.

PROOF. By Artin’s approximation theorem [Ar69, Theorem 1.6], the family \overline{X} can be defined over the Henselization of $k[t]$. From there, we descend to some k -algebra of finite type and spread out to some projective family $\overline{Y} \rightarrow C$, where C is a smooth and projective curve over k . We denote by $0 \in C$ the point such that the family over the completed ring $\widehat{\mathcal{O}}_{C,0}$ is \overline{X} . Since Shioda-supersingular K3 surfaces have potential good reduction [RS82], we may assume (after possibly replacing C by a finite flat cover) that $\overline{Y} \rightarrow C$ is a smooth and projective family of supersingular K3 surfaces.

It remains to show claim (2). We keep the notations introduced in the proof of Proposition 3.4. In particular, the Jacobian elliptic fibration on X extends to an elliptic fibration on the generic fiber $\overline{Y}_{k(C)}$, which is generically an A -torsor. Let $c \in C$ be a closed point and let \overline{Y}_c be the fiber. If we assume that $\sigma_0(\overline{Y}_c) = \sigma_0(\overline{Y}_{k(C)})$, then no class in $\text{NS}(\overline{Y}_{k(C)})$ can become p -divisible after specialization to \overline{Y}_c . Hence, the elliptic fibration on $\overline{Y}_{k(C)}$ specializes to an elliptic fibration on \overline{Y}_c , stays generically an A -torsor, and the degree- p multisection D specializes to a

degree- p multisection D_c on \overline{Y}_c . Since D is purely inseparable over the base, the same is true for its specialization D_c , and then, the arguments given in the proof of claims (2) and (3) of Proposition 3.4 show that \overline{Y}_c is related to X by a purely inseparable isogeny of height 2. \square

3.3. Jacobian elliptic fibrations on supersingular K3 surfaces. In order to be able to apply Proposition 3.4, we have to show the existence of Jacobian elliptic fibrations on supersingular K3 surfaces. For example, a supersingular K3 surface with Artin invariant $\sigma_0 = 10$ cannot possess such a fibration, for otherwise Proposition 3.4 would produce a supersingular K3 surface with $\sigma_0 = 11$, which is impossible. The next proposition shows that this is the only restriction.

Proposition 3.6. *Let X be a supersingular K3 surface with Artin invariant σ_0 in characteristic $p \geq 5$.*

- (1) *If $\sigma_0 \leq 9$, then X admits a Jacobian elliptic fibration.*
- (2) *If $\sigma_0 = 10$, then X does not admit a Jacobian elliptic fibration.*

Remark 3.7. The second assertion was already shown by Kondō and Shimada [KS12, Corollary 1.6] using different methods.

PROOF. We have shown claim (2) in the lines before this proposition.

By [RS78, Section 1], the Artin invariant σ_0 determines the Néron–Severi lattice of X up to isometry, and we denote this lattice by Λ_{p,σ_0} . To show the existence of a Jacobian elliptic fibration on X , it suffices to find an embedding of the rank 2 lattice U' with intersection matrix

$$\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$$

into Λ_{p,σ_0} . Since U' is isometric to a hyperbolic plane U , and since Λ_{p,σ_0} is a sublattice of Λ_{p,σ_0-1} for every $\sigma_0 \geq 2$, it suffices to show that $\Lambda_{p,9}$ contains U in order to establish claim (1). However, this follows immediately from the explicit classification of the lattices Λ_{p,σ_0} in [RS78, Section 1]: namely, there exists an isometry

$$\Lambda_{p,9} \cong U \oplus H_p \oplus (I(-p)^{16})_*,$$

where the other lattices are defined and explained in [RS78, Section 1]. \square

3.4. Small Characteristics. In order to keep the previous section more readable, we have avoided discussing quasi-elliptic fibrations, that is, genus-one fibrations that are not generically smooth. However, this phenomenon exists for $p \leq 3$ only, and we leave it to the reader to verify that

- (1) Proposition 3.2 and Proposition 3.4 hold in any characteristic. If $p \leq 3$, then they also remain true when replacing “(Jacobian) elliptic” by “quasi-elliptic (with a section)”.
- (2) If $p \leq 3$, then Proposition 3.6 becomes:
 - (a) No genus-one fibration on a supersingular K3 surface with $\sigma_0 = 10$ admits a section. Also, no quasi-elliptic fibration on a supersingular K3 surface with $\sigma_0 = 6$ in characteristic 3 admits a section.

- (b) Every supersingular K3 surface with $\sigma_0 \leq 9$ admits a genus-one fibration with a section.

Unfortunately, Corollary 3.5 rests on the Rudakov–Shafarevich theorem [RS82] on potential good reduction of supersingular K3 surfaces, which (currently) requires the assumption $p \geq 5$, see also Section 4.1 below.

4. MODULI SPACES

In this section, we interpret the one-dimensional families of Proposition 3.4 and Corollary 3.5 in terms of moduli spaces. In order to avoid technical difficulties, we work with moduli spaces of rigidified K3 crystals rather than moduli spaces of marked supersingular K3 surfaces. As an interesting byproduct, we find that the former spaces are related to each other by (iterated) \mathbb{P}^1 -bundles, together with a moduli interpretation of this structure. In particular, this gives a new description of these moduli spaces, see Remark 4.4.

Let N be a *supersingular K3 lattice*, that is, the Néron–Severi lattice of a supersingular K3 surface in characteristic $p \geq 5$. By [RS78, Section 1], such a lattice is determined up to isometry by p and its Artin invariant σ_0 .

Definition 4.1. Let N be a supersingular K3 lattice. An N -marked supersingular K3 surface is a K3 surface X together with an isometric embedding $N \rightarrow \mathrm{NS}(X)$.

In [Og83, Theorem (2.7)], Ogus showed the existence of a fine moduli scheme \mathcal{S}_N for N -marked supersingular K3 surfaces, and proved that it is locally of finite presentation, locally separated, and smooth of dimension $\sigma_0(N) - 1$ over \mathbb{F}_p . As explained on [Og83, p. 380], \mathcal{S}_N is almost proper, but neither of finite type nor separated over \mathbb{F}_p . (As on [Og83, p. 374], we call a scheme *almost proper*, if it satisfies the existence part of the valuative criterion for properness with DVR’s as test rings.)

A K3 crystal of rank 22 consists of a triple $(H, \langle -, - \rangle, \Phi)$, where H is free W -module of rank 22, $\langle -, - \rangle$ is a symmetric bilinear form on H , and Φ is a Frobenius-linear endomorphism of H , that satisfy the conditions of [Og79, Definition 3.1]. For example, the F -crystal associated to a K3 surface is a K3 crystal. In case H is of slope one, the K3 crystal is called *supersingular*. By the crystalline Torelli theorem [Og83, Theorem I], a Shioda-supersingular K3 surface in characteristic $p \geq 5$ is determined up to isomorphism by its supersingular K3 crystal.

In order to construct the period map, we first have to rigidify the K3 crystals: by [Og79, Definition 3.2], the *Tate-module* of a K3 crystal H is defined to be $T_H := \{x \in H : \Phi(x) = px\}$. Then, in [Og79, Theorem 3.3] it is shown that for supersingular K3 crystals, T_H is a free \mathbb{Z}_p -module of rank 22, and that the bilinear form $\langle -, - \rangle$ on H induces a non-degenerate, but non-perfect bilinear form on T_H . Moreover, an N -marking of a supersingular K3 surface induces an isometric embedding of N into the Tate-module of its associated K3 crystal, which motivates the following definition.

Definition 4.2. Let N be a supersingular K3 lattice. An N -rigidified K3 crystal is a pair $(\iota : N \rightarrow T_H, H)$, where H is a K3 crystal, and ι is an isometric embedding.

By [Og79, Proposition 4.6], there exists a moduli space \mathcal{M}_N of N -rigidified K3 crystals, which is smooth and projective of dimension $\sigma_0(N) - 1$ over \mathbb{F}_p . We refer to Remark 4.4 and the references given there for details about its geometry. As explained in [Og83, Section 3], assigning to an N -marked supersingular K3 surface its N -rigidified K3 crystal induces a morphism $\pi : \mathcal{S}_N \rightarrow \mathcal{M}_N$.

In order to get the period map, we have to equip N -rigidified K3 crystals with ample cones, and refer to [Og83, Definition 1.15] for a precise definition. There exists a moduli scheme \mathcal{P}_N for N -rigidified K3 crystals with ample cones, which is almost proper and locally of finite type over \mathbb{F}_p . Forgetting the ample cone induces a morphism $f_N : \mathcal{P}_N \rightarrow \mathcal{M}_N$, which is étale and surjective, but neither of finite type, nor separated [Og83, Proposition (1.16)]. Finally, assigning to an N -marked supersingular K3 surface its N -rigidified supersingular K3 crystal together with the ample cone arising from the ample cone of X , defines a lift of π to a morphism

$$\tilde{\pi} : \mathcal{S}_N \longrightarrow \mathcal{P}_N.$$

This is the *period map*, and it is an isomorphism by [Og83, Theorem III’].

After these preparations, we now interpret Proposition 3.4 and Corollary 3.5 in terms of moduli spaces of rigidified K3 crystals. It is likely that this result extends to moduli spaces \mathcal{S}_N of N -marked supersingular K3 surfaces, but since these are neither of finite type nor separated, the proofs and maybe even the statements would probably be rather technical and involved.

Theorem 4.3. *Let N and N_+ be the supersingular K3 lattices in characteristic $p \geq 5$ of Artin-invariants σ_0 and $\sigma_0 + 1$, respectively. Then, there exists a surjective morphism ϖ_N of moduli spaces of rigidified K3 crystals*

$$\begin{array}{c} \mathcal{M}_{N_+} \\ \sigma_N \left\downarrow \varpi_N \right. \\ \mathcal{M}_N \end{array}$$

which turns \mathcal{M}_{N_+} into a \mathbb{P}^1 -bundle over \mathcal{M}_N , and where σ_N is a section. These maps have the following moduli interpretation:

- (1) *Let X be a supersingular K3 surface with $\mathrm{NS}(X) \cong N$, and let $[X] \in \mathcal{M}_N$ be the associated K3 crystal. Then, there exists an N_+ -marked family as in Corollary 3.5*

$$\overline{Y} \rightarrow C,$$

such that the associated N_+ -rigidified K3 crystals map onto $\varpi_N^{-1}([X])$.

- (2) *Being the fiber over $0 \in C$, the surface X inherits an N_+ -marking, and the corresponding K3 crystal is $\sigma_N([X])$.*

PROOF. Let us first set up the lattices: let X be a supersingular K3 surface over an algebraically closed field k with $\mathrm{NS}(X) \cong N$. By Proposition 3.6, there exists a Jacobian elliptic fibration $X \rightarrow \mathbb{P}^1$. Let E be the class of a fiber, and Z be the zero section. Then, E and Z span a hyperbolic plane U inside N . By Proposition 3.4, there exists an elliptic supersingular K3 surface X_+ over $k[[t]]$ with special fiber X , such that the elliptic fibration extends from X to X_+ , such that there exists a

degree- p multisection D on X_+ , and such that $N_+ := \text{NS}(X_+)$ has Artin invariant $\sigma_0 + 1$. We have $D = pZ$ (see the proof of Proposition 3.4) and the sublattice of U generated by D and E is isometric to $U(p)$. In particular, the specialization homomorphism $\text{NS}(X_+) \rightarrow \text{NS}(X)$ gives rise to embeddings of lattices

$$\begin{array}{ccc} N_+ & \rightarrow & N \\ \uparrow & & \uparrow \\ U(p) & \rightarrow & U \end{array}$$

Now, we define σ_N . By [Og79, Definition 4.1], \mathcal{M}_N parametrizes N -rigidified K3 crystals, that is, pairs $(\iota : N \rightarrow T_H, H)$ as in Definition 4.2. Composing ι with $N_+ \rightarrow N$, we turn an N -rigidified K3 crystal into an N_+ -rigidified K3 crystal, which defines a morphism $\sigma_N : \mathcal{M}_N \rightarrow \mathcal{M}_{N_+}$. The image of σ_N consists precisely of those N_+ -rigidified K3 crystals $(\iota_+ : N_+ \rightarrow T_{H_+}, H_+)$ such that the class $Z = \frac{1}{p}D$ lies in the Tate-module T_{H_+} .

To define ϖ_N and to compute its fibers, it is more convenient to work with characteristic subspaces rather than rigidified K3 crystals. We refer to [Og79, Proposition 4.3] for the translation between these two points of view. We set $N_0 := pN^\vee/pN$ and $(N_+)_0 := pN_+^\vee/pN_+$, and note that these are \mathbb{F}_p -vector spaces of dimensions $2\sigma_0$ and $2\sigma_0 + 2$, respectively. The intersection forms on N and N_+ are divisible by p on pN^\vee and pN_+^\vee , and induce perfect forms on N_0 and $(N_+)_0$ after division by p , see [Og79, Proposition 3.13]. Moreover, the embedding $U(p) \subset N_+$ induces an isometry of $(N_+)_0$ with $N_0 \perp (U \otimes \mathbb{F}_p)$, where $U \otimes \mathbb{F}_p$ is generated by the classes of D and E . Tensoring the inclusion $N_+ \subset N$ with k , we obtain a map $\gamma : N_+ \otimes k \rightarrow N \otimes k$, which has a one-dimensional kernel generated by D , and whose cokernel is one-dimensional generated by $\gamma(Z)$. We thus obtain a commutative diagram of k -vector spaces

$$\begin{array}{ccc} (N_+)_0 \otimes_{\mathbb{F}_p} k & \cong & (N_0 \otimes_{\mathbb{F}_p} k) \perp (U \otimes_{\mathbb{Z}} k) \subset N_+ \otimes_{\mathbb{Z}} k \\ & & \downarrow \gamma \\ N_0 \otimes_{\mathbb{F}_p} k & \subset & N \otimes_{\mathbb{Z}} k \end{array}$$

We set $\varphi := \text{id} \otimes F_k^*$ on $N_0 \otimes k$ and $(N_+)_0 \otimes k$, where F_k denotes Frobenius. By [Og79, Definition 3.19], a *characteristic subspace* of $N_0 \otimes k$ (resp. $(N_+)_0 \otimes k$) is a k -subvector space of the form $\varphi^{-1}(K)$, where K is totally isotropic of dimension σ_0 (resp. $\sigma_0 + 1$), and $K + \varphi(K)$ is of dimension $\sigma_0 + 1$ (resp. $\sigma_0 + 2$). It is called *strictly characteristic* if it is characteristic and moreover $\sum_{i=0}^{\infty} \varphi^i(K)$ is equal to $N_0 \otimes k$ (resp. $(N_+)_0 \otimes k$). For example, if $\varphi^{-1}(K) \subset N_0 \otimes k$ is characteristic, then a straight forward computation shows that $\varphi^{-1}(\gamma^{-1}(K))$ is a characteristic subspace of $(N_+)_0 \otimes k$, but never strictly characteristic. Using [Og79, Proposition 4.3], it is not difficult to see that $\varphi^{-1}(K) \mapsto \varphi^{-1}(\gamma^{-1}(K))$ is the map σ_N in terms of characteristic subspaces. (Alternatively, if we view $N_0 \otimes k$ as a subspace of $(N_+)_0 \otimes k$, then $\varphi^{-1}(K) \mapsto \varphi^{-1}(\langle K, D \rangle)$ is equal to σ_N . This way, we see that there exists a second map $\sigma'_N : \mathcal{M}_N \rightarrow \mathcal{M}_{N_+}$ defined by $\varphi^{-1}(K) \mapsto \varphi^{-1}(\langle K, E \rangle)$.)

Let us now define ϖ_N using characteristic subspaces: for a k -subvector space K of $(N_+)_0 \otimes k$, we set $\Gamma_+(K) := \text{pr}_{N_0}(K \cap E^\perp)$, where pr_{N_0} denotes the projection

from $(N_+)_0 \otimes k$ onto $N_0 \otimes k$. A tedious, but straight forward calculation shows that if $\varphi^{-1}(K)$ is characteristic, then so is $\varphi^{-1}(\Gamma_+(K))$, and this defines a morphism $\varpi_N : \mathcal{M}_{N_+} \rightarrow \mathcal{M}_N$. By construction, we have $\Gamma_+(\gamma^{-1}(K)) = K$ for every $K \subseteq N_0 \otimes k$, which implies $\varpi_N \circ \sigma_N = \text{id}$. In particular, ϖ_N is surjective, and σ_N is a section of ϖ_N . (Remark: the map σ'_N is also a section of ϖ_N .)

To compute the fibers of ϖ_N , we fix a k -rational point of \mathcal{M}_N , that is, a characteristic subspace $\varphi^{-1}(K_0) \subset N_0 \otimes k$. Let pr_U be the projection from $(N_+)_0 \otimes k \cong (N_0 \otimes k) \perp (U \otimes k)$ onto $U \otimes k$. A straight forward computation shows that if $\varphi^{-1}(K_+) \subset (N_+)_0 \otimes k$ is characteristic, then $\text{pr}_U(K_+ \cap \varphi(K_+))$ is one-dimensional. First, this defines a surjective morphism $\mathcal{M}_{N_+} \rightarrow \mathbb{P}(U) \cong \mathbb{P}^1$. Second, it shows that $K_+ \cap \varphi(K_+) \cap (N_0 \otimes k)$ is $(\sigma_0 - 1)$ -dimensional. (Again, we view $N_0 \otimes k$ as a subspace of $(N_+)_0 \otimes k$.) In particular, if $\Gamma_+(K_+) = K_0$, then $K_+ \cap \varphi(K_+) \cap (N_0 \otimes k) = K_0 \cap \varphi(K_0)$. Thus, every characteristic subspace $K_+ \subset (N_+)_0 \otimes k$ with $\Gamma_+(K_+) = K_0$ contains the $(\sigma_0 - 1)$ -dimensional and totally isotropic subspace $K_0 \cap \varphi(K_0)$. Let $k_1, \dots, k_{\sigma_0-1}$ be a basis of $K_0 \cap \varphi(K_0)$, and let $v \in K_0$ such that $K_0 = \langle v, K_0 \cap \varphi(K_0) \rangle$ and $\varphi(K_0) = \langle \varphi(v), K_0 \cap \varphi(K_0) \rangle$. We normalize v such that $\langle v, \varphi(v) \rangle = 1$. Then, another straight forward calculation shows that $K_+ \subset (N_+)_0 \otimes k$ is characteristic with $\Gamma_+(K_+) = K_0$ if and only if either $K_+ = \langle K_0, E \rangle$ or there exists some $\lambda \in k$ such that

$$(5) \quad K_+ = \langle k_1, \dots, k_{\sigma_0-1}, v + \lambda E, v - \lambda \varphi(v) + D + \lambda E \rangle.$$

This shows that fibers of ϖ_N over geometric points are isomorphic to \mathbb{P}^1 , and it implies that ϖ_N is a conic bundle. But having a section (namely, σ_N), ϖ_N is a \mathbb{P}^1 -bundle. (Let us also note the following, which we will not need in the sequel: for a solution K_+ of $\Gamma_+(K_+) = K_0$ as in (5), we have

$$\text{pr}_U(K_+ \cap \varphi(K_+)) = \langle D - \lambda^{p+1} E \rangle,$$

which shows that there is a μ_{p+1} -ambiguity in recovering K_+ from K_0 and this projection. In particular, combining ϖ_N with the projection onto $\mathbb{P}(U)$, we obtain a finite surjective Galois morphism $\mathcal{M}_{N_+} \rightarrow \mathcal{M}_N \times \mathbb{P}^1$ with group μ_{p+1} . Its ramification locus is the union of the images of the two sections σ_N and σ'_N .)

It remains to show the moduli interpretation of ϖ_N and σ_N . For this, let X be a supersingular K3 surface with $\sigma_0(X) = \sigma_0(N)$ corresponding to a k -rational point of \mathcal{M}_N . We may choose the marking $N \cong \text{NS}(X)$ such that the embedding $U \rightarrow N$ gives rise to a Jacobian elliptic fibration on X . Then, we let $\bar{Y} \rightarrow C$ be the associated family of supersingular K3 surfaces of Corollary 3.5. Let η be the generic point of C , set $R := \mathcal{O}_{C,\eta}$, whose field of fractions is $k(C)$, and fix a uniformizer $t \in R$. The marking $N \rightarrow \text{NS}(X)$ induces a marking $N_+ \rightarrow \text{NS}(\bar{Y}_{k(C)})$ of the generic fiber, and via restriction, the whole family $\bar{Y} \rightarrow C$ becomes N_+ -marked. Then, the Chern class c_{dR} and the natural restriction maps

induce a commutative diagram

$$\begin{array}{ccccc}
\mathrm{NS}(\overline{Y}_{k(C)}) & \rightarrow & \mathrm{NS}(\overline{Y}_{k(C)}) \otimes_{\mathbb{Z}} k(C) & \xrightarrow{c_{dR}} & H_{dR}^2(\overline{Y}_{k(C)}/k(C)) \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{NS}(\overline{Y}_R) & \rightarrow & \mathrm{NS}(\overline{Y}_R) \otimes_{\mathbb{Z}} R & \xrightarrow{c_{dR}} & H_{dR}^2(\overline{Y}_R/R) \\
\downarrow & & \downarrow \gamma' & & \downarrow \\
\mathrm{NS}(X) & \rightarrow & \mathrm{NS}(X) \otimes_{\mathbb{Z}} k & \xrightarrow{c_{dR}} & H_{dR}^2(X/k)
\end{array}$$

As explained on [Og83, page 365], the characteristic subspace associated to a marked supersingular K3 surface is the kernel of c_{dR} . We identify N with $\mathrm{NS}(X)$, N_+ with $\mathrm{NS}(\overline{Y}_R)$, and let $K'_0 := \varphi^{-1}(K_0) \subset N_0 \otimes k$ be the characteristic subspace associated to X . It is not difficult to see that there exists a lift of K'_0 to an R -submodule $\tilde{K}'_0 \subset N_+ \otimes R$ of rank σ_0 that is contained in $\ker(c_{dR})$. More precisely, if k_1, \dots, k_{σ_0} is a basis of K'_0 , and $\bar{k}_i := k_i \otimes 1 \in N_0 \otimes R$, there exist lifts of the k_i to $\ker(c_{dR})$ of the form

$$\bar{k}_i + t\bar{n}_i + \alpha_i D + t\beta_i E, \quad i = 1, \dots, \sigma_0,$$

where $\bar{n}_i \in N_0 \otimes R$, and $\alpha_i, \beta_i \in R$. There is one more element in $\ker(c_{dR})$, linearly independent from these, and without loss of generality it is not divisible by t and lies in the kernel of γ' . Thus, we may choose it to be of the form

$$t\bar{n}_0 + D + t\beta E,$$

where $\bar{n}_0 \in N_0 \otimes R$ and $\beta \in R$. Since these $\sigma_0 + 1$ elements lie inside $\ker(c_{dR})$, they form a totally isotropic subspace. After some tedious computations exploiting this isotropy, we find that $\ker(c_{dR})$ contains a free R -submodule \tilde{K}'_+ of rank $\sigma_0 + 1$ generated by elements of the form

$$\begin{array}{ccc}
\bar{k}_i & & + t\mu_i E \\
t\bar{n}_0 & + D & + t\beta E
\end{array}$$

Thus, $\tilde{K}'_+ \otimes \overline{k(C)} \subset (N_+)_0 \otimes \overline{k(C)}$ is the characteristic subspace associated to $\overline{Y}_{k(C)}$. Using the explicit description, we compute $\gamma'(\tilde{K}'_+) = K'_0$, and $\Gamma_+(\tilde{K}'_+) = K'_0 \otimes_k R$. In particular, the classifying map $f_C : C \rightarrow \mathcal{M}_{N_+}$ maps to the fiber $\varpi_N^{-1}([X])$. Since this fiber is irreducible, C is a curve, and f_C is not constant, f_C is surjective onto $\varpi_N^{-1}([X])$, which establishes claim (1). The fiber over $0 \in C$ is isomorphic to X , and the N_+ -marking of $\mathrm{NS}(X)$ induced from the N_+ -marking of the family $\overline{Y} \rightarrow C$ arises via $N_+ \rightarrow N \cong \mathrm{NS}(X)$. Thus, the associated N_+ -rigidified K3 crystal is $\sigma_N([X])$, which establishes claim (2). \square

Remark 4.4. The geometry of the moduli space $\mathcal{M}_N \times_{\mathbb{F}_p} k$, where $k := \overline{\mathbb{F}_p}$, was already determined in [Og79, Examples 4.7] in the following cases

$\sigma_0(N)$	$\mathcal{M}_N \times_{\mathbb{F}_p} k$
1	$\mathrm{Spec} \, k \sqcup \mathrm{Spec} \, k$
2	$\mathbb{P}^1 \times (\mathrm{Spec} \, k \sqcup \mathrm{Spec} \, k)$
3	$(\mathbb{P}^1 \times \mathbb{P}^1) \times (\mathrm{Spec} \, k \sqcup \mathrm{Spec} \, k)$

By the previous result, $\mathcal{M}_N \times_{\mathbb{F}_p} k$ is an iterated \mathbb{P}^1 -bundle over $\mathrm{Spec} k \sqcup \mathrm{Spec} k$, and we have given a moduli interpretation for this structure. For other descriptions, we refer to [Og79, Remark 4.8] and [Og79, Theorem 3.21].

In the course of the proof we constructed a finite and surjective Galois morphism

$$\mathcal{M}_{N_+} \longrightarrow \mathcal{M}_N \times \mathbb{P}^1,$$

and saw that it is ramified over the union of two sections σ_N, σ'_N of ϖ_N . It would be interesting to pursue this further. Another interesting question is whether \mathcal{M}_N is isomorphic to $(\mathbb{P}^1)^{\sigma_0-1} \times \mathbb{F}_{p^2}$, viewed as an \mathbb{F}_p -scheme.

As a direct consequence of the previous proof and Corollary 3.5, we obtain the following result, whose proof we leave to the reader.

Corollary 4.5. *Let X be a supersingular K3 surface with Artin invariant $\sigma_0 \geq 2$ in characteristic $p \geq 5$. Then, there exists an embedding of the lattice $\mathbb{Z}D \oplus \mathbb{Z}E$ with intersection matrix*

$$\begin{pmatrix} -2p^2 & p \\ p & 0 \end{pmatrix}$$

into $\mathrm{NS}(X)$, such that

- (1) *E is the class of a fiber of a non-Jacobian elliptic fibration.*
- (2) *D is the class of a degree- p multisection, which is purely inseparable over the base of the fibration.*
- (3) *The associated Jacobian elliptic fibration is a supersingular K3 surface with Artin invariant $\sigma_0 - 1$, and is related to X by a purely inseparable isogeny of height 2.* □

4.1. Small Characteristics. Theorem 4.3 relies heavily on Ogus' articles [Og79] and [Og83]. In [Og79], the theory of supersingular K3 crystals is developed, and the assumption $p \geq 3$ is built in from the very beginning (quadratic and symplectic forms in characteristic p play an important role in this article). In [Og83], $p \geq 5$ had to be assumed because it rests on [Og79] and it needs the theorem of Rudakov–Shafarevich [RS82] potential good reduction of supersingular K3 surfaces, see also the footnote on [Og83, p. 364]. Once supersingular K3 surfaces in $p = 3$ are shown to have potential good reduction, the results in [Og83] and the results of this section will hold in $p = 3$ as well.

5. SUPERSINGULAR K3 SURFACES ARE UNIRATIONAL

In this section, we prove that supersingular K3 surfaces in characteristic $p \geq 5$ are related by purely inseparable isogenies, which is an analog of the Shioda–Inose theorem for singular K3 surfaces (Theorem 2.6). It also answers a question of Rudakov and Shafarevich from [RS78]. As a direct corollary, we deduce the Artin–Rudakov–Shafarevich–Shioda conjecture on unirationality of all supersingular K3 surfaces.

5.1. Isogenies between supersingular K3 surfaces. We now come to the main theorem of this article, which is a structure result for supersingular K3 surfaces, similar to Theorem 2.6 for singular K3 surfaces. We note that such a theorem was posed as an open question by Rudakov and Shafarevich (Question 8 at the end of [RS78]), and we refer to Section 2.2 for putting this result into perspective to a conjecture of Shafarevich about isogenies between complex K3 surfaces. Bearing all this in mind, we have:

Theorem 5.1. *Let X and X' be supersingular K3 surfaces with Artin invariants σ_0 and σ'_0 in characteristic $p \geq 5$.*

- (1) *If $\sigma_0 \leq 9$, then there exists a supersingular K3 surface with Artin invariant $\sigma_0 + 1$ that is purely inseparably isogenous of height 2 to X .*
- (2) *If $\sigma_0 \geq 2$, then there exists a supersingular K3 surface with Artin invariant $\sigma_0 - 1$ that is purely inseparably isogenous of height 2 to X .*
- (3) *There exist purely inseparable isogenies*

$$X \dashrightarrow X' \dashrightarrow X,$$

both of which are of height $2\sigma_0 + 2\sigma'_0 - 4$.

- (4) *Let E be a supersingular elliptic curve. Then, there exist isogenies*

$$\mathrm{Km}(E \times E) \dashrightarrow X \dashrightarrow \mathrm{Km}(E \times E),$$

both of which are purely inseparable of height $2\sigma_0 - 2$.

PROOF. Claim (2) follows immediately from Corollary 4.5. To show claim (1), we pick a Jacobian elliptic fibration on X , which exists by Proposition 3.6. Then, Proposition 3.4 provides us with a supersingular K3 surface with Artin invariant $\sigma_0 + 1$ that is purely inseparably isogenous of height 2 to X .

Applying the established claim (2) inductively, we obtain a purely inseparable isogeny φ of height $2\sigma_0 - 2$ from X to a supersingular K3 surface with Artin invariant $\sigma_0 = 1$. However, there exists only one such surface, namely $\mathrm{Km}(E \times E)$, where E is a supersingular elliptic curve [Og79, Corollary (7.14)]. The $(2\sigma_0 - 2)$ -fold Frobenius $X \rightarrow X$ factors through φ and we obtain claim (4).

By the established claim (4), there exists a purely inseparable isogeny $\varphi' : \mathrm{Km}(E \times E) \dashrightarrow X'$ of height $2\sigma'_0 - 2$. Then, $\varphi' \circ \varphi$ is a purely inseparable isogeny $X \dashrightarrow X'$ of height $2\sigma_0 + 2\sigma'_0 - 4$. As before, the $(2\sigma_0 + 2\sigma'_0 - 4)$ -fold Frobenius $X \rightarrow X$ factors through $\varphi' \circ \varphi$ and we obtain claim (3). \square

Remark 5.2. Naively, one might expect that K3 surfaces of Picard rank $\geq \rho$ form a codimension ρ subset inside the 20-dimensional formal moduli space. First, this subset is not algebraic: for example, polarized K3 surfaces form a countable union of divisors, and singular K3 surfaces form a countable set of points, but the naive dimension expectation is fulfilled. In this picture, one might expect that surfaces with $\rho = 22$ should not exist at all, and the fact that they come in 9-dimensional families is even more puzzling. However, by Theorem 5.3, there exists only one supersingular K3 surface in every positive characteristic up to purely inseparable isogeny. Also, the 9-dimensional moduli space is explained by the fact that these

purely inseparable isogenies come in families, see Proposition 3.4 and Theorem 4.3.

5.2. Supersingular K3 surfaces are unirational. Since Shioda [Sh77b] showed that supersingular Kummer surfaces are unirational, the previous theorem implies the conjecture of Artin, Rudakov, Shafarevich, and Shioda.

Theorem 5.3. *Supersingular K3 surfaces in characteristic $p \geq 5$ are unirational.*

PROOF. In odd characteristic, supersingular Kummer surfaces are unirational by [Sh77b, Theorem 1.1]. The assertion then follows from Theorem 5.1.(4). \square

We recall that a surface is called a *Zariski surface* if there exists a dominant, rational, and purely inseparable map of degree p from \mathbb{P}^2 onto it. Although the map from \mathbb{P}^2 onto a supersingular Kummer surface constructed by Shioda in [Sh77b] is inseparable, it is not purely inseparable. Using a different construction, Katsura [Ka87, Theorem 5.10] showed that supersingular Kummer surfaces with $\sigma_0 = 1$ in characteristic $p \not\equiv 1 \pmod{12}$ are Zariski surfaces. This strengthens Theorem 5.3, and gives a partial answer to a question of Rudakov and Shafarevich, who asked and doubted whether supersingular K3 surfaces are purely inseparably unirational (Question 6 at the end of [RS78]).

Corollary 5.4. *A supersingular K3 surface in characteristic $p \geq 5$ with $p \not\equiv 1 \pmod{12}$ is purely inseparably unirational.* \square

We remind the reader of Section 2.1, where we discussed the different notions of supersingularity for K3 surfaces and its relation to the Tate-conjecture. Now, combining Theorem 2.3 and Theorem 5.3, we obtain the following equivalence.

Theorem 5.5. *For a K3 surface X in characteristic $p \geq 5$, the following conditions are equivalent:*

- (1) *X is unirational.*
- (2) *The Picard rank of X is 22.*
- (3) *The formal Brauer group of X is of infinite height.*
- (4) *For all i , the F -crystal $H_{\text{cris}}^i(X/W)$ is of slope $i/2$.*

PROOF. If X is unirational, then its Picard rank is 22 by [Sh74, Corollary 2], which establishes (1) \Rightarrow (2). The converse direction (2) \Rightarrow (1) is Theorem 5.3. The equivalences (2) \Leftrightarrow (3) \Leftrightarrow (4) are Theorem 2.3. \square

5.3. Small Characteristics. As in Section 3.4 and Section 4.1, let us discuss what we know and do not know in characteristic $p \leq 3$.

- (1) Using quasi-elliptic fibrations, Rudakov and Shafarevich [RS78] showed that Shioda-supersingular K3 surfaces in $p = 2$ and supersingular K3 surfaces with $\sigma_0 \leq 6$ in $p = 3$ are Zariski surfaces, and thus, unirational. Therefore, the question remains whether supersingular K3 surfaces with $\sigma_0 \geq 7$ in $p = 3$ are unirational. By Proposition 3.2 together with the comments made in Section 3.4, there exists at least a 6-dimensional family of unirational K3 surfaces with $\sigma_0 = 7$ in $p = 3$.

- (2) Theorem 5.1 rests on Corollary 4.5, and we refer to Section 4.1 for details. On the other hand, quasi-elliptic K3 surfaces in $p \leq 3$ are Zariski surfaces, and thus, automatically related by purely inseparable isogenies.
- (3) The implication (1) \Rightarrow (2) of Theorem 5.5 holds in any characteristic and we discussed the converse direction above. The implication (2) \Rightarrow (3) holds in any characteristic, and its converse would follow from the Tate-conjecture for K3 surfaces, which is true in $p = 3$ by [MP13]. The equivalence (3) \Leftrightarrow (4) holds in every characteristic.

In particular (see also Section 4.1), once supersingular K3 surfaces in $p = 3$ are shown to have potential good reduction, all results of this section will hold for $p = 3$ as well.

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